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ORIGINAL RESEARCH

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New entanglement-assisted quantum MDS codes with length $n=\frac{q^2+1}{10\mu}$

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Abstract
In this work, by investigating the decomposition of the defining set of constacyclic codes, we obtain two types of q-ary entanglement-assisted quantum MDS(EAQMDS) codes with length $n=\frac{q^2+1}{10\mu}$, where m is a positive integer, q is an odd prime power such that $q=10\mu m+\nu$ or $q=10\mu m+10\mu-\nu$, and both μ and ν are odd with $10\mu=\nu^2+1$ and $\nu\geq 3$. Some of which are minimum distance achieves $\frac{q}{2}+1$ or even greater than $\frac{q}{2}+1$. Moreover, comparing the parameters with those of all known EAQMDS codes, the q-ary EAQMDS codes exhibited here are not covered in the sense that their parameters are more general than the results what have been previously known in the literature.

Keywords EAQECCs · Constacyclic codes · EAQMDS codes

Mathematics Subject Classification 94B15 · 94B65

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1 Introduction

Since the groundbreaking work with reducing the effects of decoherence for information stored in quantum memory and methods of error correction in the quantum regime and their limitations assessed in [1,2], the theory of quantum error-correcting codes(QECCs) has undergone great progress in [3–5]. Entanglement-assisted stabilizer formalism has been presented in [6]. Due to the prominent advantage of EAQECCs, many scholars constructed good EAQECCs by applying classical linear codes [7–16].

Let q be a prime power, an $[[n, k, d; c]]_q$ code denotes an EAQECC of length n and minimum distance d over $\mathbb{F}_q$, which encodes k information qubits into n-channel qubits via c pairs of maximally entangled states and can correct up to $\lfloor \frac{d-1}{2} \rfloor$ errors. Moreover, the q-ary EAQECC $[[n, k, d; c]]$ is called EAQMDS codes [17] when $k = n - 2(d - 1) + c$. Li et al. researched methods and determined a maximal design distance of BCH codes and Hermitian dual containing [18]. Later, Li et al. devised a new scheme in determining pairs of the maximally entangled states, which improved the previous calculation methods and obtained several good entanglement-assisted quantum codes [19]. Then, Chen et al. generalized the method and utilized the decomposition of the defining set of negacyclic codes and obtained four classes of EAQMDS codes with the minimum distance exceeding $q + 1$ [20]. Through the study of constacyclic codes, Chen and Zhu et al. [21] and Koroglu. [22] constructed four classes of new EAQECCs with length $n = \frac{q^2+1}{5}$ and eight new classes of EAQMDS codes, respectively.

Recently, by analyzing defining set of cyclic codes, Wang et al. presented a few new EAQMDS codes that their lengths divide q^2+1 , where q is an odd prime power in [23]. In [24], Lu et al. constructed some new quantum MDS codes with length $n = \frac{q^2+1}{a}$ and $q = 2am+t$ by using constacyclic codes and Hermitian construction. Then, Chen et al. considered the case of entanglemet-assisted quantum MDS codes with length $n = \frac{q^2+1}{\gamma}$, where $\gamma = t^2 + 1$ and t is a power of 2 [25], while they did not discuss the case that q is odd. Afterward, Chen et al. [26] constructed six classes of EAQMDS codes with the form $q = \alpha m + t$ or $\alpha m + \alpha - t$ and $n = \frac{q^2+1}{\alpha}$ by consuming 1, 5, 9 pre-shared maximally entangled states. In [27], Chen et al. constructed four classes of asymmetric quantum codes and two new families of optimal quantum convolutional codes with length $n = \frac{q^2+1}{10h}$. Inspired by the research above, we employ the method of a decomposition of the defining set of constacyclic codes to determine the number of shared pairs, and finally we derive two new families of EAQMDS codes as follows:

(1) $[[\frac{q^2+1}{10\mu}, \frac{q^2+1}{10\mu} - 2d + 6, d; 4]]_q$, where m is a positive integer, q is an odd prime power such that $q = 10\mu m + \nu$, and both μ and ν are odd with $10\mu = \nu^2 + 1$ and $\nu \geq 3$, $\frac{(\nu+1)q-\nu+1}{10\mu} + 2 \leq d \leq \frac{(3\nu-1)q+\nu+3}{10\mu}$ and d is odd.

(2) $[[\frac{q^2+1}{10\mu}, \frac{q^2+1}{10\mu} - 2d + 6, d; 4]]_q$, m is a positive integer, q is an odd prime power such that $q = 10\mu m + 10\mu - \nu$, and both μ and ν are odd with $10\mu = \nu^2 + 1$, $\frac{(\nu+1)q+\nu-1}{10\mu} + 2 \leq d \leq \frac{(3\nu-1)q-\nu-3}{10\mu}$ and d is odd.

The work is organized as follows: In Sect. 2, we review some essential knowledge of constacyclic codes and EAQECCs. In Sect. 3, we obtain two families of EAQMDS codes are constructed. We serve some comparisons of codes and conclusions in Sect. 4.

2 Preliminaries

2.1 Review of constacyclic codes

In the subsection, we will begin with some relevant concepts on constacyclic codes. More further and detailed information, the readers can refer to [15,16,20–22,24–26,28,29].

Let $\mathbb{F}_{q^2}$ be the finite field with q^2 elements , where q is a power of a prime p. A q^2-ary linear code $\mathcal{C}$ of length n is a nonempty subspace of $\mathbb{F}^n_{q^2}$. For any element $a \in \mathbb{F}_{q^2}$, the conjugate of a denoted by $\overline{a} = a^q$.

Given two vectors $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1})$ and $\mathbf{y} = (y_0, y_1, \ldots, y_{n-1}) \in \mathbb{F}^n_{q^2}$, their Hermitian inner product is defined as

$$\langle \mathbf{x}, \mathbf{y} \rangle = x_0\overline{y}_0 + x_1\overline{y}_1 + \cdots + x_{n-1}\overline{y}_{n-1} \in \mathbb{F}_{q^2}.$$

For a q^2-ary linear code $\mathcal{C}$ of length n, the following code

$$\mathcal{C}^{\perp h} = \{\mathbf{x} \in \mathbb{F}^n_{q^2} | \langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{0} \ for\ all\ \mathbf{y} \in \mathcal{C}\}.$$

is called Hermitian dual code of $\mathcal{C}$. If $\mathcal{C} \subseteq \mathcal{C}^{\perp h}$,then code $\mathcal{C}$ is said to be Hermitian self-orthogonal, and the code $\mathcal{C}$ is said to be Hermitian self-dual if $\mathcal{C} = \mathcal{C}^{\perp h}$.

A q^2-ary linear code $\mathcal{C}$ of length n is a η-constacyclic if there exists $\eta \in \mathbb{F}^*_{q^2}$, such that for any codeword $\mathbf{x} = (x_0, x_1, \ldots, x_{n-1}) \in \mathbb{F}^n_{q^2}$, the η-constacyclic shift of $\mathbf{x}$, $\phi_\eta(\mathbf{x}) = (\eta x_{n-1}, x_0, x_1, \ldots, x_{n-2})$, is also in $\mathcal{C}$. Customarily, each codeword $\mathbf{!} = (\omega_0, \omega_1, \ldots, \omega_{n-1})$ in $\mathcal{C}$ corresponds to its polynomial representation $\omega(x) = \omega_0 + \omega_1 x + \cdots + \omega_{n-1}x^{n-1}$. It is well known that a q^2-ary linear code of length n is an η-constacyclic code if and only if $\mathcal{C}$ is precisely an ideal of the quotient ring $\mathbb{F}_{q^2}[x]/\langle x^n - \eta\rangle$. Moreover, $\mathcal{C}$ is generated by a monic divisor $g(x)$ of $x^n - \eta$. The polynomial $g(x)$ is called the generator polynomial of the code $\mathcal{C}$.

We assume that n and q are coprime, and $\eta \in \mathbb{F}^*_{q^2}$, let r be the order of η and δ be a primitive rn-th root of unity in some extension field of $\mathbb{F}_{q^2}$ such that $\delta^n = \eta$. So the root of $x^n - \eta$ are exactly the element δ^{1+ri}.

Let $\Omega = \{j = 1 + ir | 0 \leq i \leq n-1\}$. For each $j \in \Omega$, let $\mathbf{C}_j$ be the q^2-cyclotomic coset modulo rn containing j.

$$\mathbf{C}_j = \{jq^{2l} \mod rn | 0 \leq l \leq m_j - 1\},$$

where m_j is the smallest positive integer such that $jq^{2m_j} \equiv j \mod rn$. Each $\mathbf{C}_j$ corresponds to an irreducible divisor of $x^n - \eta$. Let $\mathcal{C} = \langle g(x) \rangle$ be a q^2-ary η-constacyclic code of length n. Then, the defining set of $\mathcal{C}$ is the $Z = \{j \in \Omega | g(\delta^j) =$

0}. Clearly, $dim(\mathcal{C}) = n - |Z|$, where $|Z|$ is called the cardinality of the set Z. Note that $\mathcal{C}^{\perp h} = \{z \in \Omega | -qz \bmod rn \notin Z\}$(see Ref. [16]).

The following conclusions are of great significance to the construction of EAQMDS codes.

Proposition 1 [30,31] (The BCH bound for constacyclic codes) *Assume q and n are coprime. Let $\mathcal{C} = \langle g(x) \rangle$ be a q^2-ary η-constacyclic code of length n with the defining set $Z = \{1 + ri | l \leq i \leq l + d - 2\}$, where δ is a primitive root of unity. Then, the minimum distance of $\mathcal{C}$ is at least d.*

Proposition 2 [16] *Let r be a positive divisor of $q + 1$ and $\eta \in \mathbb{F}_{q^2}^*$ be of order r. If $\mathcal{C}$ be a q^2-ary η-constacyclic code of length n with defining set $Z \subseteq \Omega$, then $\mathcal{C}$ contains its Hermitian dual code if and only if $Z \cap Z^{-q} = \emptyset$, where $Z^{-q} = \{-qz \bmod rn | z \in Z\}$.*

2.2 Review of EAQECCs

In the subsection, we recall some background knowledge and results of EAQECCs. The detailed information on EAQECCs can be found in [6,8,18–21].

Let H be an $(n - k) \times n$ parity check matrix of $\mathcal{C}$ over $\mathbb{F}_{q^2}$. Then, $\mathcal{C}^{\perp h}$ has an $(n - k) \times n$ generator matrix $H^\dagger$, where $H^\dagger$ is the conjugate transpose matrix of H over $\mathbb{F}_{q^2}$.

Next, we review the Hermitian method, which is important for us to construct EAQECCs from classical linear codes.

The following proposition is about the Singleton bound of classical linear codes.

Proposition 3 (Singleton bound) [29] *If a q-ary $[n,k,d]$ linear code exists, then $k \leq n - d + 1$. If $k = n - d + 1$, then $\mathcal{C}$ is called an MDS codes.*

Theorem 1 [6,8,19] *If $\mathcal{C}$ is a q^2-ary $[n, k, d]$ classical code and H is its parity check matrix, then there exist EAQECCs with parameters $[[n, 2k - n + c, d; c]]_q$, where $c = rank(HH^\dagger)$.*

Theorem 2 [6] *Assume that $\mathcal{C} = [[n, k, d; c]]_q$ is an entanglement-assisted quantum code, where $d \leq \frac{n+2}{2}$, then $\mathcal{C}$ satisfies the entanglement-assisted Singleton bound $n + c - k \geq 2(d - 1)$. If $\mathcal{C}$ satisfies the equality $n + c - k = 2(d - 1)$ for $d \leq \frac{n+2}{2}$, then it is called an entanglement-assisted quantum MDS codes.*

From Proposition 1, Theorem 1 and 2, we can arrive at the following collary.

Corollary 1 *Suppose $\mathcal{C}$ be a q^2-ary constacyclic code of length n with defining set Z. Suppose $c = Z \cap (-qZ)$, where $-qZ = \{-qz \mod rn | z \in Z\}$. If $\mathcal{C}$ has parameters $[n, n - |Z|, d]_{q^2}$, then there exists an EAQECC with parameters $[[n, n - 2|Z| + c, d; c]]_q$.*

3 New EAQMDS codes of length $n = \frac{q^2+1}{10\mu}$

In this section, we use η-constacyclic code of length $n = \frac{q^2+1}{10\mu}$ to construct some new EAQMDS codes, where q is an odd prime power, $\eta \in \mathbb{F}_q^*$ with order $q+1$, which is different from the code obtained from [25]. Moreover, we obtain parameters with the even number of entangled states $c = 4$, it needs to be emphasized that the codes we give here are not covered by [26].

Note that m is a positive integer, q is an odd prime power. To avoid repetition, it is not described in this subsection.

Lemma 1 *Let $n = \frac{q^2+1}{10\mu}$, $q = 10\mu m + \nu$ or $q = 10\mu m + 10\mu - \nu$, and both μ and ν are odd with $10\mu = \nu^2 + 1$ and $\nu \geq 3$. Assume that $s = \frac{(q+10\mu+1)n}{2}$ and $\zeta = \frac{q^2-q}{2}$. Then the q^2-cyclotomic cosets modulo $(q+1)n$ are: $\mathbf{C}_s = \{s\}$, and $\mathbf{C}_{\zeta+(q+1)i} = \{\zeta + (q+1)i, \zeta - (q+1)(i-1)\}$ for $1 \leq i \leq \frac{n-1}{2}$.*

Proof $1 + (q+1)j = s$ when $j = \frac{n+q-1}{2}$. Hence, s must be in Ω. Note that $sq^2 = s(q^2+1-1) \equiv s \mod (q+1)n$, it follows that $\mathbf{C}_s = \{s\}$. Since

$$\begin{aligned}
&[\zeta + (q+1)i]q^2 \\
&= \left[s - \frac{(q+1)(n+1)}{2} + (q+1)i\right]q^2 \\
&\equiv s - (q+1)\left[\frac{n+1}{2}(q^2+1) - \frac{n+1}{2} - i(q^2+1) + i\right] \\
&\equiv s + (q+1)\left(\frac{n+1}{2} - i\right) \\
&\equiv \zeta + (q+1)(n+1-i) \\
&\equiv \zeta - (q+1)(i-1) \mod (q+1)n.
\end{aligned}$$

and

$$\begin{aligned}
&[\zeta - (q+1)(i-1)]q^2 \\
&= \left[s - \frac{(q+1)(n+1)}{2} - (q+1)(i-1)\right]q^2 \\
&\equiv s - (q+1)\left[\frac{n+1}{2}(q^2+1) - \frac{n+1}{2} + (i-1)(q^2+1) - i + 1\right] \\
&\equiv s + (q+1)\left(\frac{n+1}{2} + i - 1\right) \\
&\equiv \zeta + (q+1)(n+1+i-1) \\
&\equiv \zeta + (q+1)i \mod (q+1)n.
\end{aligned}$$

Therefore $\mathbf{C}_{\zeta+(q+1)i} = \{\zeta + (q+1)i, \zeta - (q+1)(i-1)\}$. It then remains to prove that $\mathbf{C}_{\zeta+(q+1)i} = \{\zeta + (q+1)i, \zeta - (q+1)(i-1)\}$ is disjoint for $1 \leq i \leq \frac{n-1}{2}$.

We assume that there exist two integers l and j such that $\mathbf{C}_{\zeta+(q+1)l} = \mathbf{C}_{\zeta-(q+1)(j-1)}$, where $1 \leq l \neq j \leq \frac{n-1}{2}$, then we have $\zeta+(q+1)l \equiv (\zeta+(q+1)j)q^{2k}$ for $k \in \{0, 1\}$ mod $(q+1)n$.

When $k = 0$, then $\zeta+(q+1)l \equiv \zeta+(q+1)j \mod (q+1)n$, i.e. $l = j$, however, $0 \leq l \neq j \leq \frac{n-1}{2}$, which gives a contradiction.

If $k = 1$, then $\zeta+(q+1)l \equiv \zeta-(q+1)(j-1) \mod (q+1)n$, i.e. $l + j \equiv 1$ mod $(q+1)n$, however, $2 \leq l + j \leq n-1$, which also gives a contradiction. ◻

Theorem 3 *Let $n = \frac{q^2+1}{10\mu}$, $q = 10\mu m+\nu$, and both μ and ν are odd with $10\mu = \nu^2+1$ and $\nu \geq 3$. Assume that $s = \frac{(q+10\mu+1)n}{2}$ and $\zeta = \frac{q^2-q}{2}$. If $\mathcal{C}$ is a q^2-ary η-constacyclic code of n with defining set $Z = \bigcup_{i=1}^{\delta} \mathbf{C}_{\zeta+(q+1)i}$, where $1 \leq \delta \leq \frac{(\nu+1)(q-\nu)}{20\mu}$. Then $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$.*

Proof Suppose that $\mathcal{C}$ does not contain its Hermitian dual code. Then, by Proposition 2, we just need to prove that $Z \cap -qZ \neq \emptyset$. Hence, there exist two integers l and j, where $1 \leq l, j \leq \frac{(\nu+1)(q-\nu)}{20\mu}$, such that $\zeta+(q+1)i \equiv -q(\zeta+(q+1)j)q^{2k}$ mod $(q+1)n$ for $k \in \{0, 1\}$. They conflict with the following facts.

Case I When $k = 0$, then $\zeta+(q+1)l \equiv -q(\zeta+(q+1)j) \mod (q+1)n$, i.e., $0 \equiv qj + l - \frac{q+1}{2} \mod n$, for $1 \leq l, j \leq \frac{(\nu+1)(q-\nu)}{20\mu}$.

(i) When $1 \leq j \leq \frac{q-\nu}{10\mu}$, since $0 < \frac{q+1}{2} = q+1-\frac{q+1}{2} \leq qj+l-\frac{q+1}{2} \leq q\frac{q-\nu}{10\mu} + \frac{(\nu+1)(q-\nu)}{20\mu} - \frac{q+1}{2} = \frac{2q^2-(10\mu+\nu-1)q-\nu+1-20\mu}{20\mu} < n$.
This is a contradiction, because $0 \equiv qj + l - \frac{q+1}{2} \mod n$.

(ii) When $\frac{q-\nu}{10\mu}+1 \leq j \leq \frac{2(q-\nu)}{10\mu}$, let $j' = j - \frac{q-\nu}{10\mu}$ for $1 \leq j' \leq \frac{q-\nu}{10\mu}$. Then $0 \equiv q(j'+\frac{q-\nu}{10\mu})+l-\frac{q+1}{2} \mod n$, i.e., $0 \equiv qj'+l-\frac{(2\nu+10\mu)q+10\mu+2}{20\mu} \mod n$, since

$$
\begin{aligned}
0 &< \frac{(5\mu-\nu)q+5\mu-1}{10\mu} \\
&= q+1-\frac{(2\nu+10\mu)q+10\mu+2}{20\mu} \\
&\leq qj'+l-\frac{(2\nu+10\mu)q+10\mu+2}{20\mu} \\
&\leq q\frac{q-\nu}{10\mu}+\frac{(\nu+1)(q-\nu)}{20\mu}-\frac{(2\nu+10\mu)q+10\mu+2}{20\mu} \\
&= \frac{2q^2-(10\mu+3\nu-1)q-20\mu-\nu-1}{20\mu} < n
\end{aligned}
$$

This is a contradiction, because $0 \equiv qj'+l-\frac{(2\nu+10\mu)q+10\mu+2}{20\mu} \mod n$.

(iii) When $\frac{(\epsilon-1)(q-\nu)}{10\mu}+1 \leq j \leq \frac{\epsilon(q-\nu)}{10\mu}$, where $3 \leq \epsilon \leq \frac{\nu+1}{2}$. Let $j' = j - \frac{(\epsilon-1)(q-\nu)}{10\mu}$ for $1 \leq j' \leq \frac{q-\nu}{10\mu}$. Then $0 \equiv q(j'+\frac{(\epsilon-1)(q-\nu)}{10\mu})+l-\frac{q+1}{2} \mod n$, i.e., $0 \equiv$

$qj' + l - \frac{(10\mu+2(\epsilon-1)\nu)q+10\mu+2(\epsilon-1)}{20\mu} \mod n$, since

$$\begin{aligned}
0 &< \frac{(\nu+1)q + 10\mu - \nu + 1}{20\mu} \\
&\le \frac{(10\mu - 2(\epsilon-1)\nu)q + 10\mu - 2(\epsilon-1)}{20\mu} \\
&= q + 1 - \frac{(10\mu + 2(\epsilon-1)\nu)q + 10\mu + 2(\epsilon-1)}{20\mu} \\
&\le qj' + l - \frac{(10\mu + 2(\epsilon-1)\nu)q + 10\mu + 2(\epsilon-1)}{20\mu} \\
&\le q\frac{q-\nu}{10\mu} + \frac{(\nu+1)(q-\nu)}{20\mu} \\
&\quad - \frac{(10\mu + 2(\epsilon-1)\nu)q + 10\mu + 2(\epsilon-1)}{20\mu} \\
&= \frac{2q^2 - (10\mu + (2\epsilon-1)\nu - 1)q - 20\mu - \nu + 3 - 2\epsilon}{20\mu} \\
&\le \frac{2q^2 - (10\mu + 5\nu - 1)q - 20\mu - \nu - 3}{20\mu} < n
\end{aligned}$$

This is a contradiction, because $0 \equiv qj' + l - \frac{(10\mu+2(\epsilon-1))q+10\mu+2(\epsilon-1)}{20\mu} \mod n$.

Case II When $k = 1$, then $\zeta + (q+1)l \equiv -(\zeta + (q+1)j)q^3 \mod (q+1)n$, i.e., $0 \equiv qj - l - \frac{q-1}{2} \mod n$, for $1 \le l, j \le \frac{(\nu+1)(q-\nu)}{20\mu}$.

(i) When $1 \le j \le \frac{q-\nu}{10\mu}$, since

$$\begin{aligned}
0 &< \frac{(10\mu - \nu - 1)q + \nu - 1 + 20\mu}{20\mu} \\
&= q - \frac{(\nu+1)(q-\nu)}{20\mu} - \frac{q-1}{2} \\
&\le qj - l - \frac{q-1}{2} \\
&\le q\frac{q-\nu}{10\mu} - 1 - \frac{q-1}{2} \\
&\le \frac{2q^2 - (10\mu + 2\nu)q - 10\mu}{20\mu} < n
\end{aligned}$$

This is a contradiction, because $0 \equiv qj - l - \frac{q-1}{2} \mod n$.

(ii) When $\frac{q-\nu}{10\mu} + 1 \le j \le \frac{2(q-\nu)}{10\mu}$, let $j' = j - \frac{q-\nu}{10\mu}$ for $1 \le j' \le \frac{q-\nu}{10\mu}$. Then $0 \equiv q(j' + \frac{q-\nu}{10\mu}) - l - \frac{q-1}{2} \mod n$, i.e., $0 \equiv qj' - l - \frac{(2\nu+10\mu)q-10\mu+2}{20\mu} \mod n$, since

$$\begin{aligned}
0 &< \frac{(10\mu - 3\nu - 1)q + 20\mu + \nu - 3}{20\mu} \\
&= q - \frac{(\nu+1)(q-\nu)}{20\mu} - \frac{(2\nu + 10\mu)q - 10\mu + 2}{20\mu} \\
&\leq qj' - l - \frac{(2\nu + 10\mu)q - 10\mu + 2}{20\mu} \\
&\leq q\frac{q-\nu}{10\mu} - 1 - \frac{(2\nu + 10\mu)q - 10\mu + 2}{20\mu} \\
&= \frac{q^2 - (5\mu + 2\nu)q - 5\mu - 1}{10\mu} < n
\end{aligned}$$

This is a contradiction, because $0 \equiv qj' - l - \frac{(2\nu+10\mu)q-10\mu+2}{20\mu} \mod n$.

(iii) When $\frac{(\epsilon-1)(q-\nu)}{10\mu} + 1 \leq j \leq \frac{\epsilon(q-\nu)}{10\mu}$, where $3 \leq \epsilon \leq \frac{\nu+1}{2}$. Let $j' = j - \frac{(\epsilon-1)(q-\nu)}{10\mu}$ for $1 \leq j' \leq \frac{q-\nu}{10\mu}$. Then $0 \equiv q(j' + \frac{(\epsilon-1)(q-\nu)}{10\mu}) - l - \frac{q-1}{2} \mod n$, i.e., $0 \equiv qj' - l - \frac{(10\mu+2(\epsilon-1)\nu)q-10\mu+2(\epsilon-1)}{20\mu} \mod n$, since

$$\begin{aligned}
0 < 1 &\leq \frac{(10\mu - (2\epsilon - 1)\nu - 1)q + 20\mu + \nu - 2\epsilon + 1}{20\mu} \\
&= q - \frac{(\nu+1)(q-\nu)}{20\mu} - \frac{(10\mu + 2(\epsilon-1)\nu)q - 10\mu + 2(\epsilon - 1)}{20\mu} \\
&\leq qj' - l - \frac{(10\mu + 2(\epsilon-1)\nu)q - 10\mu + 2(\epsilon - 1)}{20\mu} \\
&\leq q\frac{q-\nu}{10\mu} - 1 - \frac{(10\mu + 2(\epsilon-1)\nu)q - 10\mu + 2(\epsilon - 1)}{20\mu} \\
&= \frac{q^2 - (5\mu + \epsilon\nu)q - 5\mu - \epsilon + 1}{10\mu} \\
&\leq \frac{q^2 - (5\mu + 3\nu)q - 5\mu - 2}{10\mu} < n
\end{aligned}$$

This is a contradiction, because $0 \equiv qj' - l - \frac{(10\mu+2(\epsilon-1)\nu)q-10\mu+2(\epsilon-1)}{20\mu} \mod n$.

Finally, we conclude that $Z \cap -qZ = \emptyset$. □

Theorem 4 *Let $n = \frac{q^2+1}{10\mu}$, $q = 10\mu m + \nu$, and both μ and ν are odd with $10\mu = \nu^2 + 1$ and $\nu \geq 3$. Then there exist entanglement-assisted quantum MDS codes with parameters $[[n, n - 2d + 6, d; 4]]_q$, for each odd integer d with $\frac{(\nu+1)q-\nu+1}{10\mu} + 2 \leq d \leq \frac{(3\nu-1)q+\nu+3}{10\mu}$.*

Proof Consider the q^2-ary η-constacyclic code $\mathcal{C}$ with defining set $Z = \bigcup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} \mathbf{C}_{\zeta+(q+1)i}$, where $1 \leq \lambda \leq \frac{(\nu-1)q+\nu+1}{10\mu}$. It is easy to see that $|Z| = \frac{(\nu+1)(q-\nu)}{10\mu} + 2\lambda$, where $|Z|$ represents the cardinality of Z. By Propositions 1 and 3 , we can derive $\mathcal{C}$

is a q^2-ary MDS constacyclic code with parameters

$$\left[\frac{q^2+1}{10\mu}, \frac{q^2+1}{10\mu} - \frac{(\nu+1)(q-\nu)}{10\mu} - 2\lambda, \frac{(\nu+1)(q-\nu)}{10\mu} + 2\lambda + 1\right]_{q^2}.$$

Hence, we obtain the following result.

$$\begin{aligned}
&Z \cap (-qZ)\\
&= \left(\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right) \cup \left(\cup_{i=\frac{(\nu+1)(q-\nu)}{20\mu}+1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right)\right)\\
&\quad \cap \left(-q\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right) \cup -q\left(\cup_{i=\frac{(\nu+1)(q-\nu)}{20\mu}+1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right)\right)\\
&= \left(\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right) \cap -q\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right)\right)\\
&\quad \cup \left(\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right) \cap -q\left(\cup_{i=\frac{(\nu+1)(q-\nu)}{20\mu}+1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right)\right)\\
&\quad \cup \left(\left(\cup_{i=\frac{(\nu+1)(q-\nu)}{20\mu}+1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right) \cap -q\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right)\right)\\
&\quad \cup \left(\left(\cup_{i=\frac{(\nu+1)(q-\nu)}{20\mu}+1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right) \cup -q\left(\cup_{i=\frac{(\nu+1)(q-\nu)}{20\mu}+1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right)\right)\\
&= C_{\zeta+(q+1)\frac{(\nu+1)q+10\mu+1}{20\mu}} \cup C_{\zeta+(q+1)\frac{(\nu-1)q+10\mu+\nu+1}{20\mu}} \qquad \bigstar
\end{aligned}$$

By Theorem 3, we obtain

$$\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right) \cap -q\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right) = \emptyset.$$

Now, we need to testify that

$$\left(\cup_{i=\frac{(\nu+1)(q-\nu)}{20\mu}+1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right) \cap -q\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right) = C_{\zeta+(q+1)\frac{(\nu+1)q+10\mu-\nu+1}{20\mu}}$$
$$\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right) \cap -q\left(\cup_{i=\frac{(\nu+1)(q-\nu)}{20\mu}+1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right) = C_{\zeta+(q+1)\frac{(\nu-1)q+10\mu+\nu+1}{20\mu}}$$

and

$$\left(\cup_{i=\frac{(\nu+1)(q-\nu)}{20\mu}+1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right) \cup -q\left(\cup_{i=\frac{(\nu+1)(q-\nu)}{20\mu}+1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right) = \emptyset.$$

We first show that

$$\left(\cup_{i=\frac{(\nu+1)(q-\nu)}{20\mu}+1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right) \cap -q\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right) = C_{\zeta+(q+1)\frac{(\nu+1)q+10\mu-\nu+1}{20\mu}}$$

Clearly, $-q(C_{\zeta+(q+1)(\frac{(\nu+1)(q-\nu)}{20\mu}+1)}) = -q(C_{\zeta+(q+1)\frac{(\nu+1)q+10\mu+1}{20\mu}})$
$= C_{\zeta+(q+1)\frac{(\nu-1)q+10\mu+\nu+1}{20\mu}}$ Therefore,

$$\begin{aligned}
&\left(\cup_{i=\frac{(\nu+1)(q-\nu)}{20\mu}+1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right) \cap -q\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right)\\
&= \left(C_{\zeta+(q+1)\frac{(\nu+1)q+10\mu-\nu+1}{20\mu}} \cup \left(\cup_{\frac{(\nu+1)(q-\nu)}{20\mu}+2}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right)\right) \cap\\
&\quad - q\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right)\\
&= \left(C_{\zeta+(q+1)\frac{(\nu+1)q+10\mu-\nu+1}{20\mu}} \cap -q\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right)\right.\\
&\quad \left.\cup\left(\left(\cup_{\frac{(\nu+1)(q-\nu)}{20\mu}+2}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right) \cap -q\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right)\right)\right)\\
&= C_{\zeta+(q+1)\frac{(\nu+1)q+10\mu-\nu+1}{20\mu}}
\end{aligned}$$

From Theorem 3,

$$C_{\zeta+(q+1)\frac{(\nu+1)q+10\mu-\nu+1}{20\mu}} \cap -q(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}) = C_{\zeta+(q+1)\frac{(\nu+1)q+10\mu-\nu+1}{20\mu}}$$

We claim that

$$\left(\cup_{\frac{(\nu+1)(q-\nu)}{20\mu}+2}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right) \cap -q\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right) = \emptyset$$

for $2 \le \lambda \le \frac{(\nu-1)q+\nu+1}{10\mu}$. If $(\cup_{\frac{(\nu+1)(q-\nu)}{20\mu}+2}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}) \cap -q(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}) \neq \emptyset$ i.e.,

$$\left(\cup_{i=2}^{\lambda} C_{\zeta+(q+1)(i+\frac{(\nu+1)(q-\nu)}{20\mu})}\right) \cap -q\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right) \neq \emptyset,$$

for $2 \le \lambda \le \frac{(\nu-1)q+\nu+1}{10\mu}$. Then there exist integers l and j, where $2 \le l \le \frac{(\nu-1)q+\nu+1}{10\mu}$ and $1 \le j \le \frac{(\nu+1)(q-\nu)}{20\mu}$, such that

$$\zeta + (q+1)\left(l + \frac{(\nu+1)(q-\nu)}{20\mu}\right) \equiv -q(\zeta + (q+1)j)q^{2k} \mod (q+1)n,$$

for some $k \in \{0, 1\}$. They contradict with the following facts.

Case I When $k = 0$, $\zeta + (q+1)(l + \frac{(\nu+1)(q-\nu)}{20\mu}) \equiv -q(\zeta + (q+1)j) \mod (q+1)n$, i.e.,

$$0 \equiv qj + l - \frac{(10\mu - \nu - 1)q + \nu - 1 + 20\mu}{20\mu} \mod n,$$

for $2 \le l \le \frac{(\nu-1)q+\nu+1}{10\mu}$ and $1 \le j \le \frac{(\nu+1)(q-\nu)}{20\mu}$.

(i) When $1 \le j \le \frac{q-\nu}{10\mu}$, since

$$\begin{aligned} 0 &< \frac{(10\mu+\nu+1)q+1+20\mu-\nu}{20\mu} \\ &= q+2-\frac{(10\mu-\nu-1)q+\nu-1+20\mu}{20\mu} \\ &\le qj+l-\frac{(10\mu-\nu-1)q+\nu-1+20\mu}{20\mu} \\ &\le q\frac{q-\nu}{10\mu}+\frac{(\nu-1)q+\nu+1}{10\mu} \\ &\quad -\frac{(10\mu-\nu-1)q+\nu-1+20\mu}{20\mu} \\ &= \frac{2q^2-(10\mu-\nu+1)q+\nu+3-20\mu}{20\mu} < n \end{aligned}$$

This is a contradiction, because $0 \equiv l+qj-\frac{(10\mu-\nu-1)q+20\mu+\nu-1}{20\mu} \mod n$.

(ii) When $\frac{q-\nu}{10\mu}+1 \le j \le \frac{2(q-\nu)}{10\mu}$, let $j' = j-\frac{q-\nu}{10\mu}$ for $1 \le j' \le \frac{q-\nu}{10\mu}$. Then we have $0 \equiv q(j'+\frac{q-\nu}{10\mu})+l-\frac{(10\mu-\nu-1)q+\nu-1+20\mu}{20\mu} \mod n$, i.e., $0 \equiv qj'+l-\frac{(10\mu+\nu-1)q+20\mu+\nu+1}{20\mu} \mod n$, since

$$\begin{aligned} 0 &< \frac{(10\mu-\nu+1)q+20\mu-\nu+1}{20\mu} \\ &= q+2-\frac{(10\mu+\nu-1)q+20\mu+\nu+1}{20\mu} \\ &\le qj'+l-\frac{(10\mu+\nu-1)q+20\mu+\nu+1}{20\mu} \\ &\le q\frac{q-\nu}{10\mu}+\frac{(\nu-1)q+\nu+1}{10\mu}-\frac{(\nu+10\mu-1)q+20\mu+\nu+1}{20\mu} \\ &= \frac{2q^2-(10\mu+\nu+1)q+\nu+1-20\mu}{20\mu} < n \end{aligned}$$

This is a contradiction, because $0 \equiv qj'+l-\frac{(10\mu+\nu-1)q+20\mu+\nu+1}{20\mu} \mod n$.

(iii) When $\frac{(\epsilon-1)(q-\nu)}{10\mu}+1 \le j \le \frac{\epsilon(q-\nu)}{10\mu}$, where $3 \le \epsilon \le \frac{\nu+1}{2}$. Let $j' = j-\frac{(\epsilon-1)(q-\nu)}{10\mu}$ for $1 \le j' \le \frac{q-\nu}{10\mu}$. Then we have $0 \equiv q(j'+\frac{(\epsilon-1)(q-\nu)}{10\mu})+l-\frac{(10\mu-\nu-1)q+\nu-1+20\mu}{20\mu}$

mod n, i.e., $0 \equiv qj' + l - \frac{(10\mu-1+(2\epsilon-3)\nu)q+20\mu+\nu+2\epsilon-3}{20\mu} \mod n$, since

$$\begin{aligned} 0 &< \tfrac{(\nu+1)q-\nu+1+10\mu}{10\mu} \\ &\le \tfrac{(10\mu+1-(2\epsilon-3)\nu)q+20\mu-\nu-2\epsilon+3}{20\mu} \\ &= q+2-\tfrac{(10\mu-1+(2\epsilon-3)\nu)q+20\mu+\nu+2\epsilon-3}{20\mu} \\ &\le qj'+l-\tfrac{(10\mu-1+(2\epsilon-3)\nu)q+20\mu+\nu+2\epsilon-3}{20\mu} \\ &\le q\tfrac{q-\nu}{10\mu}+\tfrac{(\nu-1)q+\nu+1}{10\mu} \\ &\quad -\tfrac{(10\mu-1+(2\epsilon-3)\nu)q+20\mu+\nu+2\epsilon-3}{20\mu} \\ &= \tfrac{2q^2-(10\mu+(2\epsilon-3)\nu+1)q-20\mu+\nu+5-2\epsilon}{20\mu} \\ &\le \tfrac{2q^2-(10\mu+3\nu+1)q-20\mu+\nu-1}{20\mu} < n \end{aligned}$$

This is a contradiction, because $0 \equiv qj' + l - \frac{(10\mu-1+(2\epsilon-3)\nu)q+20\mu+\nu+2\epsilon-3}{20\mu}$ mod n.

Case II When $k = 1$, $\zeta + (q+1)(l + \frac{(\nu+1)(q-\nu)}{20\mu}) \equiv -(\zeta + (q+1)j)q^3 \mod (q+1)n$, which is equivalent to

$$qj \equiv l + \frac{(10\mu+\nu+1)q-\nu+1-20\mu}{20\mu} \mod n,$$

for $2 \le l \le \frac{(\nu-1)q+\nu+1}{10\mu}$ and $1 \le j \le \frac{(\nu+1)(q-\nu)}{20\mu}$.

(i) When $1 \le j \le \frac{q-\nu}{10\mu}$, since

$$\begin{aligned} 0 &< \frac{(10\mu+\nu+1)q-\nu+1+20\mu}{20\mu} \\ &= 2 + \frac{(10\mu+\nu+1)q-\nu+1-20\mu}{20\mu} \\ &\le l + \frac{(10\mu+\nu+1)q-\nu+1-20\mu}{20\mu} \\ &\le \frac{(\nu-1)q+\nu+1}{10\mu} + \frac{(10\mu+\nu+1)q-\nu+1-20\mu}{20\mu} \\ &= \frac{(10\mu+3\nu-1)q+\nu+3-20\mu}{20\mu} < q \end{aligned}$$

This is a contradiction, because $q \le qj \le \frac{q^2-\nu q}{10\mu}$.

(ii) When $\frac{q-\nu}{10\mu}+1 \le j \le \frac{2(q-\nu)}{10\mu}$, let $j' = j - \frac{q-\nu}{10\mu}$ for $1 \le j' \le \frac{q-\nu}{10\mu}$. We have $q(j' + \frac{q-\nu}{10\mu}) \equiv l + \frac{(10\mu+\nu+1)q-\nu+1-20\mu}{20\mu} \mod n$, i.e., $qj' \equiv l + \frac{(10\mu+3\nu+1)q-\nu+3-20\mu}{20\mu}$

mod n, since

$$\begin{aligned} 0 &< \frac{(10\mu+3\nu+1)q-\nu+3+20\mu}{20\mu} \\ &\le l+\frac{(10\mu+3\nu+1)q-\nu+3-20\mu}{20\mu} \\ &\le \frac{(\nu-1)q+\nu+1}{10\mu}+\frac{(10\mu+3\nu+1)q-\nu+3-20\mu}{20\mu} \\ &= \frac{(10\mu+5\nu-1)q+\nu+5-20\mu}{20\mu} < q \end{aligned}$$

This is a contradiction, because $q \le qj' \le \frac{q^2-\nu q}{10\mu}$.

(iii) When $\frac{(\epsilon-1)(q-\nu)}{10\mu}+1 \le j \le \frac{\epsilon(q-\nu)}{10\mu}$, where $3 \le \epsilon \le \frac{\nu+1}{2}$. Let $j' = j-\frac{(\epsilon-1)(q-\nu)}{10\mu}$ for $1 \le j' \le \frac{q-\nu}{10\mu}$. Then we have $q(j'+\frac{(\epsilon-1)(q-\nu)}{10\mu}) \equiv l+\frac{(10\mu+\nu+1)q-\nu+1-20\mu}{20\mu}$ mod n, i.e., $l \equiv qj' - \frac{(10\mu+(2\epsilon-1)\nu+1)q-\nu-1+2\epsilon-20\mu}{20\mu}$ mod n, we have

$$\begin{aligned} 0 < 1 &\le \frac{(10\mu-(2\epsilon-1)\nu-1)q+\nu+1-2\epsilon+20\mu}{20\mu} \\ &\le q-\frac{(10\mu+(2\epsilon-1)\nu+1)q-\nu-1+2\epsilon-20\mu}{20\mu} \\ &\le qj'-\frac{(10\mu+(2\epsilon-1)\nu+1)q-\nu-1+2\epsilon-20\mu}{20\mu} \\ &\le q\frac{q-\nu}{10\mu}-\frac{(10\mu+(2\epsilon-1)\nu+1)q-\nu-1+2\epsilon-20\mu}{20\mu} \\ &= \frac{2q^2-(10\mu+(2\epsilon+1)\nu+1)q+\nu+1-2\epsilon+20\mu}{20\mu} \\ &\le \frac{2q^2-(10\mu+7\nu+1)q+\nu-5+20\mu}{20\mu} < n \end{aligned}$$

This is a contradiction, because $2 \le l \le \frac{(\nu-1)q+\nu+1}{10\mu}$.

From the above discussion, we have

$$\left(\cup_{i=\frac{(\nu+1)(q-\nu)}{20\mu}+1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right) \cap -q\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right) = C_{\zeta+(q+1)\frac{(\nu+1)q+10\mu-\nu+1}{20\mu}},$$

for $1 \le \lambda \le \frac{(\nu-1)q+\nu+1}{10\mu}$.

Secondly, we show that

$$\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right) \cap -q\left(\cup_{i=\frac{(\nu+1)(q-\nu)}{20\mu}+1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right) = C_{\zeta+(q+1)\frac{(\nu-1)q+10\mu+\nu+1}{20\mu}}.$$

Since

$$
\begin{aligned}
&-q\left(\left(\cup_{i=\frac{(\nu+1)(q-\nu)}{20\mu}+1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right)\cap -q\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right)\right)\\
&= -qC_{\zeta+(q+1)\frac{(\nu+1)q+10\mu-\nu+1}{20\mu}}\\
&= C_{\zeta+(q+1)\frac{(\nu-1)q+10\mu+\nu+1}{20\mu}}
\end{aligned}
$$

Hence,

$$
\left(\cup_{i=1}^{\frac{(\nu+1)(q-\nu)}{20\mu}} C_{\zeta+(q+1)i}\right)\cap -q\left(\cup_{i=\frac{(\nu+1)(q-\nu)}{20\mu}+1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right) = C_{\zeta+(q+1)\frac{(\nu-1)q+10\mu+\nu+1}{20\mu}}.
$$

Finally, we show that

$$
\left(\cup_{i=\frac{(\nu+1)(q-\nu)}{20\mu}+1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right)\cup -q\left(\cup_{i=\frac{(\nu+1)(q-\nu)}{20\mu}+1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}\right) = \emptyset,
$$

for $1 \leq \lambda \leq \frac{(\nu-1)q+\nu+1}{10\mu}$. If $(\cup_{i=\frac{(\nu+1)(q-\nu)}{20\mu}+1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i})\cup -q(\cup_{i=\frac{(\nu+1)(q-\nu)}{20\mu}+1}^{\frac{(\nu+1)(q-\nu)}{20\mu}+\lambda} C_{\zeta+(q+1)i}) \neq \emptyset$ i.e.,

$$
\left(\cup_{i=1}^{\lambda} C_{\zeta+(q+1)(i+\frac{(\nu+1)(q-\nu)}{20\mu})}\right)\cap -q\left(\cup_{i=1}^{\lambda} C_{\zeta+(q+1)(i+\frac{(\nu+1)(q-\nu)}{20\mu})}\right) \neq \emptyset,
$$

for $1 \leq \lambda \leq \frac{(\nu-1)q+\nu+1}{10\mu}$. Then there exist integers l and j, where $1 \leq l, j \leq \frac{(\nu-1)q+\nu+1}{10\mu}$, such that

$$
\begin{aligned}
&\zeta+(q+1)\left(l+\frac{(\nu+1)(q-\nu)}{20\mu}\right)\\
&\equiv -q\left(\zeta+(q+1)\left(j+\frac{(\nu+1)(q-\nu)}{20\mu}\right)\right)q^{2k} \mod (q+1)n,
\end{aligned}
$$

for some $k \in \{0, 1\}$. They contradict with the following facts.

Case I When $k=0$, $\zeta+(q+1)(l+\frac{(\nu+1)(q-\nu)}{20\mu}) \equiv -q(\zeta+(q+1)(j+\frac{(\nu+1)(q-\nu)}{20\mu}))$ $\mod (q+1)n$, i.e.,

$$
0 \equiv qj+l-\frac{(10\mu-1)q+\nu+10\mu}{10\mu} \mod n,
$$

for $1 \leq l, j \leq \frac{(\nu-1)q+\nu+1}{10\mu}$.

(i) When $1 \leq j \leq \frac{q-\nu}{10\mu}$, since

$$\begin{aligned}
0 &< \frac{q-\nu}{10\mu} \\
&= q+1-\frac{(10\mu-1)q+\nu+10\mu}{10\mu} \\
&\leq qj+l-\frac{(10\mu-1)q+\nu+10\mu}{10\mu} \\
&\leq q\frac{q-\nu}{10\mu}+\frac{(\nu-1)q+\nu+1}{10\mu}-\frac{(10\mu-1)q+\nu+10\mu}{10\mu} \\
&= \frac{q^2-10\mu q+1-10\mu}{10\mu} < n
\end{aligned}$$

This is a contradiction, because $0 \equiv qj+l-\frac{(10\mu-1)q+\nu+10\mu}{10\mu} \mod n$.

(ii) When $\frac{q-\nu}{10\mu}+1 \leq j \leq \frac{2(q-\nu)}{10\mu}$, let $j' = j-\frac{q-\nu}{10\mu}$ for $1 \leq j' \leq \frac{q-\nu}{10\mu}$. We have $0 \equiv q(j'+\frac{q-\nu}{10\mu})+l-\frac{(10\mu-1)q+\nu+10\mu}{10\mu} \mod n$, i.e., $\frac{(10\mu+\nu-1)q+\nu+1+10\mu}{10\mu} \equiv qj'+l \mod n$, since

$$\begin{aligned}
0 &< q+1 \\
&\leq qj'+l \\
&\leq q\frac{q-\nu}{10\mu}+\frac{(\nu-1)q+\nu+1}{10\mu} \\
&= \frac{q^2-q+\nu+1}{10\mu} < n
\end{aligned}$$

This is a contradiction, because $\frac{(10\mu+\nu-1)q+\nu+1+10\mu}{10\mu} \equiv qj'+l \mod n$.

(iii) When $\frac{(\epsilon-1)(q-\nu)}{10\mu}+1 \leq j \leq \frac{\epsilon(q-\nu)}{10\mu}$, where $3 \leq \epsilon \leq \nu-2$. Let $j' = j-\frac{(\epsilon-1)(q-\nu)}{10\mu}$ for $1 \leq j' \leq \frac{q-\nu}{10\mu}$. Then we have $0 \equiv q(j'+\frac{(\epsilon-1)(q-\nu)}{10\mu})+l-\frac{(10\mu-1)q+\nu+10\mu}{10h} \mod n$, i.e., $0 \equiv qj'+l-\frac{(10\mu-1+(\epsilon-1)\nu)q+10\mu+\nu+\epsilon-1}{10\mu} \mod n$.
When $j'=1$, i.e., $j = \frac{(\epsilon-1)(q-\nu)}{10\mu}+1 > 1$, since

$$\begin{aligned}
0 &< \frac{\nu q+10\mu+1}{10\mu} \\
&\leq \frac{((\epsilon-2)\nu+10\mu)q+10\mu+\epsilon-2}{10\mu}-q \\
&= \frac{(10\mu-1+(\epsilon-1)\nu)q+10\mu+\nu+\epsilon-1}{10\mu}-\frac{(\nu-1)q+\nu+1}{10\mu}-q \\
&\leq \frac{(10\mu-1+(\epsilon-1)\nu)q+10\mu+\nu+\epsilon-1}{10\mu}-l-q
\end{aligned}$$

$$\begin{aligned}&\le \frac{(10\mu-1+(\epsilon-1)\nu)q+10\mu+\nu+\epsilon-1}{10\mu}-1-q\\&=\frac{(10\mu-1+(\epsilon-1)\nu)q+\nu+\epsilon-1}{10\mu}-q\\&\le \frac{(10\mu-3\nu-2)q+2\nu-3}{10\mu}<n\end{aligned}$$

This is a contradiction, because $q \equiv \frac{(10\mu-1+(\epsilon-1)\nu)q+10\mu+\nu+\epsilon-1}{10\mu} - l \mod n$.
When $2 \le j' \le \frac{q-\nu}{10\mu}$, since

$$\begin{aligned}0&<\frac{(3\nu+2)q-2\nu+3}{10\mu}\\&\le \frac{(10\mu+1-(\epsilon-1)\nu)q-\nu+1-\epsilon}{10\mu}\\&=2q+1-\frac{(10\mu-1+(\epsilon-1)\nu)q+10\mu+\nu+\epsilon-1}{10\mu}\\&\le qj'+l-\frac{(10\mu-1+(\epsilon-1)\nu)q+10\mu+\nu+\epsilon-1}{10\mu}\\&\le q\frac{q-\nu}{10\mu}+\frac{(\nu-1)q+\nu+1}{10\mu}-\frac{(10\mu-1+(\epsilon-1)\nu)q+10\mu+\nu+\epsilon-1}{10\mu}\\&=\frac{q^2-((\epsilon-1)\nu+10\mu)q-10\mu-\epsilon+2}{10\mu}\\&\le \frac{q^2-(10\mu+2\nu)q-10\mu-1}{10\mu}<n\end{aligned}$$

This is a contradiction, because $0 \equiv qj'+l-\frac{(10\mu-1+(\epsilon-1)\nu)q+10\mu+\nu+\epsilon-1}{10\mu} \mod n$.

(iv) When $\frac{(\nu-2)(q-\nu)}{10\mu}+1 \le j \le \frac{(\nu-1)q+\nu+1}{10\mu}$. Let $j' = j - \frac{(\nu-2)(q-\nu)}{10\mu}$ for $1 \le j' \le \frac{q-\nu}{10\mu}+1$. Then we have $0 \equiv q(j'+\frac{(\nu-2)(q-\nu)}{10\mu})+l-\frac{(10\mu-1)q+\nu+10\mu}{10\mu} \mod n$, i.e., $0 \equiv qj'+l-\frac{2(10\mu-1-\nu)q+10\mu+2\nu-2}{10\mu} \mod n$.
When $j'=1$, since

$$\begin{aligned}0&<\frac{(10\mu-3\nu-1)q+10\mu+\nu-1}{10\mu}\\&=q+1-\frac{2(10\mu-1-\nu)q+10\mu+2\nu-2}{10\mu}\\&\le qj'+l-\frac{2(10\mu-1-\nu)q+10\mu+2\nu-2}{10\mu}\end{aligned}$$

$$
\begin{aligned}
&\leq q+\frac{(\nu-1)q+\nu+1}{10\mu}-\frac{2(10\mu-1-\nu)q+10\mu+2\nu-2}{10\mu}\\
&=\frac{(10\mu-2\nu-2)q+2\nu+2}{10\mu}<n
\end{aligned}
$$

This is a contradiction, because $q \equiv \frac{(10\mu-1+(\epsilon-1)\nu)q+10\mu+\nu+\epsilon-1}{10\mu}-l \mod n$.
When $2 \leq j' \leq \frac{q-\nu}{10\mu}+1$, then

$$
\begin{aligned}
0 &< \frac{(2\nu+2)q-2\nu+2}{10\mu}\\
&=2q+1-\frac{2(10\mu-1-\nu)q+10\mu+2\nu-2}{10h}\\
&\leq qj'+l-\frac{2(10\mu-1-\nu)q+10\mu+2\nu-2}{10\mu}\\
&\leq q\left(\frac{q-\nu}{10\mu}+1\right)+\frac{(\nu-1)q+\nu+1}{10\mu}-\frac{2(10\mu-1-\nu)q+10\mu+2\nu-2}{10\mu}\\
&=\frac{q^2-(20\mu-2\nu-1)q-10\mu-\nu+2}{10\mu}<n
\end{aligned}
$$

This is a contradiction, because $0 \equiv qj'+l-\frac{2(10\mu-1-\nu)q+10\mu+2\nu-2}{10\mu} \mod n$.

Case II When $k=1$, $\zeta+(q+1)(l+\frac{(\nu+1)(q-\nu)}{20\mu}) \equiv -q(\zeta+(q+1)(j+\frac{(\nu+1)(q-\nu)}{20\mu}))q^2 \mod (q+1)n$, i.e., $1 \leq l, j \leq \frac{(\nu-1)q+\nu+1}{10\mu}$.
When $j=1$, since

$$
\begin{aligned}
0 &< \frac{\nu q+1}{10\mu}\\
&=1+\frac{(10\mu+\nu)q+1-10\mu}{10\mu}-q\\
&\leq l+\frac{(10\mu+\nu)q+1-10\mu}{10\mu}-q\\
&\leq \frac{(\nu-1)q+\nu+1}{10\mu}+\frac{(10\mu+\nu)q+1-10\mu}{10\mu}-q\\
&=\frac{(2\nu-1)q+\nu+2-10\mu}{10\mu}<n
\end{aligned}
$$

This is a contradiction, because $qj \equiv l+\frac{(10\mu+\nu)q+1-10\mu}{10\mu} \mod n$.

(i) When $2 \leq j \leq \frac{q-\nu}{10\mu}$ (when $j=1, q=10\mu+\nu$, which we have discussed), since

$$\begin{aligned}0 &< \frac{(10\mu-2\nu+1)q-\nu+10\mu-2}{10\mu}\\ &\leq 2q-\frac{(\nu-1)q+\nu+1}{10\mu}-\frac{(10\mu+\nu)q+1-10\mu}{10\mu}\\ &\leq qj-l-\frac{(10\mu+\nu)q+1-10\mu}{10\mu}\\ &\leq q\frac{q-\nu}{10\mu}-1-\frac{(10\mu+\nu)q+1-10\mu}{10\mu}\\ &=\frac{q^2-(10\mu+2\nu)q-1}{10\mu}<n\end{aligned}$$

This is a contradiction, because $qj \equiv l+\frac{(10\mu+\nu)q+1-10\mu}{10\mu} \mod n$.

(ii) When $\frac{q-\nu}{10\mu}+1 \leq j \leq \frac{2(q-\nu)}{10\mu}$, let $j'=j-\frac{q-\nu}{10\mu}$ for $1 \leq j' \leq \frac{q-\nu}{10\mu}$. Then we have $q(j'+\frac{q-\nu}{10\mu}) \equiv l+\frac{(10\mu+\nu)q+1-10\mu}{10\mu} \mod n$, i.e., $qj' \equiv l+\frac{(10\mu+2\nu)q+2-10\mu}{10\mu} \mod n$.
When $j'=1$, $j=\frac{q-\nu}{10\mu}+1$, since

$$\begin{aligned}0 &< \frac{2\nu q+2}{10\mu}\\ &=1+\frac{(10\mu+2\nu)q+2-10\mu}{10\mu}-q\\ &\leq l+\frac{(10\mu+2\nu)q+2-10\mu}{10\mu}-q\\ &\leq \frac{(\nu-1)q+\nu+1}{10\mu}+\frac{(10\mu+2\nu)q+2-10\mu}{10\mu}-q\\ &=\frac{(3\nu-1)q+\nu+3-10\mu}{10\mu}<n\end{aligned}$$

This is a contradiction, because $q \equiv l+\frac{(10\mu+2\nu)q+2-10\mu}{10\mu} \mod n$.
When $2 \leq j' \leq \frac{q-\nu}{10\mu}$, i.e., $\frac{q-\nu}{10\mu}+1 \leq j \leq \frac{2(q-\nu)}{10\mu}$,we have

$$\begin{aligned}0 &< \frac{(10\mu-3\nu+1)q-\nu-3+10\mu}{10\mu}\\ &=2q-\frac{(\nu-1)q+\nu+1}{10\mu}-\frac{(10\mu+2\nu)q+2-10\mu}{10\mu}\\ &\leq qj'-l-\frac{(10\mu+2\nu)q+2-10\mu}{10\mu}\end{aligned}$$

$$\begin{aligned} &\le q\frac{q-\nu}{10\mu}-1-\frac{(10\mu+2\nu)q+2-10\mu}{10\mu}\\ &=\frac{q^2-(10\mu+3\nu)q-2}{10\mu}<n \end{aligned}$$

This is a contradiction, because $qj' \equiv l+\frac{(10\mu+2\nu)q+2-10\mu}{10\mu} \mod n$.

(iii) When $\frac{(\epsilon-1)(q-\nu)}{10\mu}+1 \le j \le \frac{\epsilon(q-\nu)}{10\mu}$, where $3 \le \epsilon \le \nu-2$. Let $j'=j-\frac{(\epsilon-1)(q-\nu)}{10\mu}$ for $1 \le j' \le \frac{q-\nu}{10\mu}$. Then we have $q(j'+\frac{(\epsilon-1)(q-\nu)}{10\mu}) \equiv l+\frac{(10\mu+\nu)q+1-10\mu}{20\mu} \mod n$, i.e., $0 \equiv qj'-l-\frac{(10\mu+\epsilon\nu)q+\epsilon-10\mu}{10\mu} \mod n$.
When $j'=1$, i.e., $j=\frac{(\epsilon-1)(q-\nu)}{10\mu}+1$, since

$$\begin{aligned} 0 &< \frac{3\nu q+3}{10\mu}\\ &\le \frac{(\epsilon\nu+10\mu)q+\epsilon}{10\mu}-q\\ &=1+\frac{(10\mu+\epsilon\nu)q+\epsilon-10\mu}{10\mu}-q\\ &\le l+\frac{(10\mu+\epsilon\nu)q+\epsilon-10\mu}{10\mu}-q\\ &\le \frac{(\nu-1)q+\nu+1}{10\mu}+\frac{(10\mu+\epsilon\nu)q+\epsilon-10\mu}{10\mu}-q\\ &=\frac{(10\mu-1+(\epsilon+1)\nu)q+\nu+1+\epsilon-10\mu}{10\mu}-q\\ &\le \frac{(10\mu-\nu-2)q+2\nu-1-10\mu}{10\mu}<n \end{aligned}$$

This is a contradiction, because $qj' \equiv l+\frac{(10\mu+\epsilon\nu)q+\epsilon-10\mu}{10\mu} \mod n$.
When $2 \le j' \le \frac{q-\nu}{10\mu}$, since

$$\begin{aligned} 0 &< \frac{(\nu+2)q-2\nu+1+10\mu}{10\mu}\\ &\le \frac{(10\mu+1-(\epsilon+1)\nu)q-\nu-1-\epsilon+10\mu}{10\mu}\\ &=2q-\frac{(\nu-1)q+\nu+1}{10\mu}-\frac{(10\mu+\epsilon\nu)q+\epsilon-10\mu}{10\mu}\\ &\le qj'-l-\frac{(10\mu+\epsilon\nu)q+\epsilon-10\mu}{10\mu}\\ &=q\frac{q-\nu}{10\mu}-1-\frac{(10\mu+\epsilon\nu)q+\epsilon-10\mu}{10\mu} \end{aligned}$$

$$= \frac{q^2 - (10\mu + (\epsilon+1)\nu)q - \epsilon}{10\mu}$$
$$\le \frac{q^2 - (10\mu + 4)q - 3}{10\mu} < n$$

This is a contradiction, because $0 \equiv qj' - l - \frac{(10\mu+\epsilon\nu)q+\epsilon-10\mu}{10\mu} \mod n$.

(iv) When $\frac{(\nu-2)(q-\nu)}{10\mu} + 1 \le j \le \frac{(\nu-1)q+\nu+1}{10\mu}$. Let $j' = j - \frac{(\nu-2)(q-\nu)}{10\mu}$ for $1 \le j' \le \frac{q-\nu}{10\mu} + 1$, we have $q(j' + \frac{(\nu-2)(q-\nu)}{10\mu}) \equiv l + \frac{(10\mu+\nu)q+2-20\mu}{10\mu} \mod n$, i.e., $0 \equiv qj' - l - \frac{(20\mu-1-\nu)q+\nu-20\mu}{10\mu} \mod n$.

When $j' = 1$, i.e., $j = \frac{(\nu-2)(q-\nu)}{10\mu} + 1$, since

$$0 < \frac{(10\mu - 1 - \nu)q + \nu - 10\mu}{10\mu}$$
$$= 1 + \frac{(20\mu - 1 - \nu)q + \nu - 20\mu}{10\mu} - q$$
$$\le l + \frac{(20\mu - 1 - \nu)q + \nu - 20\mu}{10\mu} - q$$
$$\le \frac{(\nu-1)q + \nu + 1}{10\mu} + \frac{(20\mu - 1 - \nu)q + \nu - 20\mu}{10\mu} - q$$
$$= \frac{(10\mu - 2)q + 2\nu + 1 - 20\mu}{10\mu} < n$$

This is a contradiction, because $qj' \equiv l + \frac{(20\mu+1-\nu)q+\nu-20\mu}{10\mu} \mod n$.

When $2 \le j' \le \frac{q-\nu}{10\mu} + 1$, since

$$0 < \frac{20\mu - 2\nu - 1}{10\mu}$$
$$= 2q - \frac{(\nu-1)q + \nu + 1}{10\mu} - \frac{(20\mu + 1 - \nu)q + \nu - 20\mu}{10\mu}$$
$$\le qj' - l - \frac{(20\mu + 1 - \nu)q + \nu - 20\mu}{10\mu}$$
$$\le q\left(\frac{q-\nu}{10\mu} + 1\right) - 1 - \frac{(20\mu + 1 - \nu)q + \nu - 20\mu}{10\mu}$$
$$= \frac{q^2 - (10\mu + 1)q - \nu + 10\mu}{10\mu} < n$$

This is a contradiction, because $0 \equiv qj' - l - \frac{(20\mu+1-\nu)q+\nu-20\mu}{10\mu} \mod n$.

From Corollary 1, there exist entanglement-assisted quantum codes with parameters $[[\frac{q^2+1}{10\mu}, \frac{q^2+1}{10\mu} - 2d + 6, d; 4]]_q$, where $\frac{(\nu-1)(q-\nu)}{10\mu} + 3 \le d \le \frac{(3\nu-1)q+\nu+3}{10\mu}$ is odd. □

Table 1 Sample parameters of EAQMDS codes constructed from Theorem 4

q	n	$[[n, k, d; c]]_q$
257	1321	$[[1321, 1241, 43; 4]]_{257}$
257	1321	$[[1321, 1237, 45; 4]]_{257}$
257	1321	$[[1321, 1233, 47; 4]]_{257}$
257	1321	$[[1321, 1229, 49; 4]]_{257}$
257	1321	$[[1321, 1225, 51; 4]]_{257}$
...	...	...
257	1321	$[[1321, 1137, 95; 4]]_{257}$
257	1321	$[[1321, 1133, 97; 4]]_{257}$
257	1321	$[[1321, 1129, 99; 4]]_{257}$
257	1321	$[[1321, 1125, 101; 4]]_{257}$
257	1321	$[[1321, 1121, 103; 4]]_{257}$

Remark 1 Taking $\mu = 1, \nu = 3$, i.e., $q = 10m + 3$, we can obtain a quantum MDS codes with parameters $[[\frac{q^2+1}{10}, \frac{q^2+1}{10} - 2d + 6, d; 4]]_q$, where $\frac{2q+9}{5} \le d \le \frac{4q+3}{5}$, which is the quantum MDS codes got in [23].

Example 1 We show some parameters of EAQMDS codes derived from Theorem 4 in Table 1.

For the case that $q = 10\mu m + 10\mu - \nu$, and both μ and ν are odd with $10\mu = \nu^2 + 1$ and $\nu \ge 3$, similarly to Theorem 3, we can obtain the following Theorem 5. Moreover, on the basis of the Theorem 5, we can obtain the following entanglement-assisted quantum MDS codes in Theorem 6, the proof of which is analogous to the proof of Theorem 4.

Theorem 5 *Let $n = \frac{q^2+1}{10\mu}$, where m is a positive integer, q is an odd prime power such that $q = 10\mu m + 10\mu - \nu$, and both μ and ν are odd with $10\mu = \nu^2 + 1$ and $\nu \ge 3$. Assume that $s = \frac{(q+10\mu+1)n}{2}$ and $\zeta = \frac{q^2-q}{2}$. If $\mathcal{C}$ is a q^2-ary η-constacyclic code of n with defining set $Z = \bigcup_{i=1}^{\delta} \mathbf{C}_{\zeta+(q+1)i}$, where $1 \le \delta \le \frac{(\nu+1)(q-\nu)+2\nu-2}{20\mu}$. Then $\mathcal{C}^{\perp_h} \subseteq \mathcal{C}$.*

Theorem 6 *Let $n = \frac{q^2+1}{10\mu}$, where $q = 10\mu m + 10\mu - \nu$, and both μ and ν are odd with $10\mu = \nu^2 + 1$ and $\nu \ge 3$. Then there exist entanglement-assisted quantum MDS codes with parameters $[[n, n - 2d + 6, d; 4]]_q$, for each odd integer d with $\frac{(\nu+1)q+\nu-1}{10\mu} + 2 \le d \le \frac{(3\nu-1)q-\nu-3}{10\mu}$.*

Remark 2 Taking $\mu = 1, \nu = 3$, i.e., $q = 10m + 7$, we can obtain a quantum MDS codes with parameters $[[\frac{q^2+1}{10}, \frac{q^2+1}{10} - 2d + 6, d; 4]]_q$, where $\frac{2q+11}{5} \le d \le \frac{4q-3}{5}$, which is the quantum MDS codes got in [23].

Example 2 We show some parameters of EAQMDS codes derived from Theorem 6 in Table 3

Table 2 Comparisons of EAQMDS codes between ours and others

Range of parameters	d	$[[n, k, d; c]]$	References
$q = 10m + 3$	$\frac{2q+9}{5} \leq d \leq \frac{4q+3}{5}$ *is odd*	$[[\frac{q^2+1}{10}, \frac{q^2+1}{10} - 2d + 6, d; 4]]_q$	[23]
$q = 10m + 7$	$\frac{2q+11}{5} \leq d \leq \frac{4q-3}{5}$ *is odd*	$[[\frac{q^2+1}{10}, \frac{q^2+1}{10} - 2d + 6, d; 4]]_q$	[23]
$q = t^e > 4$ with $e \equiv 1 \mod 4$ and $\gamma = t^2 + 1$ with t is a power of 2	$\frac{(t+1)q-t+1+2\gamma}{\gamma} \leq d \leq \frac{(3t-1)q+t+3}{\gamma}$ *and d is odd*	$[[\frac{q^2+1}{\gamma}, \frac{q^2+1}{\gamma} - 2d + 6, d; 4]]_q$	[25]
$q = t^e$ with $e \equiv 3 \mod 4$ and $\gamma = t^2 + 1$ with t is a power of 2	$\frac{(t+1)q+2\gamma+t-1}{\gamma} \leq d \leq \frac{(3t-1)q-t-3}{\gamma}$ *and d is odd*	$[[\frac{q^2+1}{\gamma}, \frac{q^2+1}{\gamma} - 2d + 6, d; 4]]_q$	[25]
q is an odd prime power with the form $q = \alpha m + t$, m is a positive integer, α and $t \geq 2$ are positive integers such that $\alpha = t^2 + 1$	$\frac{2tq+2+2\alpha}{\alpha} \leq d \leq \frac{(2(t+1)q-2(t-1)}{\alpha}$ *and d is even*	$[[\frac{q^2+1}{\alpha}, \frac{q^2+1}{\alpha} - 2d + 7, d; 5]]_q$	[26]
q is an odd prime power with the form $q = \alpha m + \alpha - t$, m is a positive integer, α and $t \geq 2$ are positive integers such that $\alpha = t^2 + 1$	$\frac{2tq-2+2\alpha}{\alpha} \leq d \leq \frac{(2(t+1)q+2(t-1)}{\alpha}$ *and d is even*	$[[\frac{q^2+1}{\alpha}, \frac{q^2+1}{\alpha} - 2d + 7, d; 5]]_q$	[26]

Table 2 continued

Range of parameters	d	$[[n, k, d; c]]$	References
q is an odd prime power with the form $q = 10\mu m + \nu$, m is a positive integer, μ and ν both are odd such that $10\mu = \nu^2 + 1$	$\frac{(\nu+1)q-\nu+1}{10\mu} + 2 \leq d \leq \frac{(3\nu-1)q+\nu+3}{10\mu}$ *and d is odd*	$[[\frac{q^2+1}{10\mu}, \frac{q^2+1}{10\mu} - 2d + 6, d; 4]]_q$	*Theorem* 4
q is an odd prime power with the form $q = 10\mu m + 10\mu - \nu$, m is a positive integer, μ and ν both are odd such that $10\mu = \nu^2 + 1$	$\frac{(\nu+1)q+\nu-1}{10\mu} + 2 \leq d \leq \frac{(3\nu-1)q-\nu-3}{10\mu}$ *and d is odd*	$[[\frac{q^2+1}{10\mu}, \frac{q^2+1}{10\mu} - 2d + 6, d; 4]]_q$	*Theorem* 6

Table 3 Sample parameters of EAQMDS codes constructed from Theorem 6

q	n	$[[n, k, d; c]]_q$
293	1717	$[[1717, 1625, 49; 4]]_{293}$
293	1717	$[[1717, 1621, 51; 4]]_{293}$
293	1717	$[[1717, 1617, 53; 4]]_{293}$
293	1717	$[[1717, 1613, 55; 4]]_{293}$
293	1717	$[[1717, 1609, 57; 4]]_{293}$
...	...	...
293	1717	$[[1717, 1505, 109; 4]]_{293}$
293	1717	$[[1717, 1501, 111; 4]]_{293}$
293	1717	$[[1717, 1497, 113; 4]]_{293}$
293	1717	$[[1717, 1493, 115; 4]]_{293}$
293	1717	$[[1717, 1489, 117; 4]]_{293}$

4 Conclusions and code comparisons

In the last section, we obtain two families EAQMDS codes q^2-ary constacyclic codes of length $n = \frac{q^2+1}{10\mu}$, where q is the odd prime such that $q = 10\mu m + \nu$ or $q = 10\mu m + 10\mu - \nu$.

From Table 2, the lengths of EAQMDS codes we constructed here are different from [25], since $q = t^e$ and $\gamma = t^2 + 1$ in [25], where t is a power of 2 . Compared with EAQMDS codes in [26], by consuming fewer pairs of maximally entangled states, we get new EAQMDS codes with odd minimum distance. According to Remark 1 and Remark 2, EAQMDS codes constructed from [23] were encompassed by Theorem 4 and Theorem 5 in this paper, and our EAQMDS codes are general, Table 2 gives the detailed comparisons.

More clearly, when μ becomes bigger, it is going to be harder for us to get greater minimum distance d of entanglement-assisted quantum MDS codes. In the future work, it would be meaningful to get more EAQMDS codes with large minimum distance by applying constacyclic codes.

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